Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER Journal of Combinatorial Theory, Series B 90 (2004) 233–255

Journal of Combinatorial Theory Series B

http://www.elsevier.com/locate/jctb

Weakly distance-regular digraphs

F. Comellas,[a] M.A. Fiol,[a] J. Gimbert,[b] and M. Mitjana[c]

[a] *Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Jordi Girona 1-3, Mòdul C3, Campus Nord, 08034 Barcelona, Spain*
[b] *Departament de Matemàtica, Universitat de Lleida, Jaume II 69, 25005 Lleida, Spain*
[c] *Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Gregorio Marañon 44, 08028 Barcelona, Spain*

Received 23 May 2001

Abstract

We introduce the concept of weakly distance-regular digraph and study some of its basic properties. In particular, the (standard) distance-regular digraphs, introduced by Damerell, turn out to be those weakly distance-regular digraphs which have a normal adjacency matrix. As happens in the case of distance-regular graphs, the study is greatly facilitated by a family of orthogonal polynomials called the distance polynomials. For instance, these polynomials are used to derive the spectrum of a weakly distance-regular digraph. Some examples of these digraphs, such as the butterfly and the cycle prefix digraph which are interesting for their applications, are analyzed in the light of the developed theory. Also, some new constructions involving the line digraph and other techniques are presented.

Keywords: Distance-regular digraph; Adjacency matrix; Spectrum; Orthogonal polynomials; Butterfly digraph; Cycle prefix digraph

1. Introduction

Let us first introduce some basic notation (see [4,17]) and discuss the relevant background. Let $\Gamma = (V, E)$ be a (strongly) connected digraph with order $N := |V|$ and diameter D. We will denote by $\mathrm{dist}(u, v)$ the distance from vertex u to vertex v. Notice that $\mathrm{dist}(u, v)$ and $\mathrm{dist}(v, u)$ may not be equal since we are considering oriented graphs. For any fixed integer $0 \leqslant k \leqslant D$, we will denote by $\Gamma_k^+(v)$ (respectively, $\Gamma_k^-(v)$) the set of vertices at distance k from v (respectively, the set of

E-mail address: fiol@mat.upc.es (M.A. Fiol).

0095-8956/$ - see front matter
doi:10.1016/j.jctb.2003.07.003

vertices from which v is at distance k). Sometimes it is written, for short, $\Gamma^+(v)$ or $\Gamma^-(v)$ instead of $\Gamma_1^+(v)$ or $\Gamma_1^-(v)$, respectively. Thus, the *out-degree* and *in-degree* of v are $\delta^+(v) := |\Gamma^+(v)|$ and $\delta^-(v) := |\Gamma^-(v)|$; and the digraph Γ is (Δ-)*regular* if $\delta^+(v) = \delta^-(v) = \Delta$ for every $v \in V$.

1.1. Distance-transitivity and distance-regularity

The concept of a distance-regular digraph was introduced by Damerell [9] in the late 1970s. He defined distance-regular digraphs by using a regularity type condition concerning the cardinality of some vertex subsets. More precisely, a digraph Γ with diameter D is *distance-regular* if, for any pair of vertices $u, v \in V$ such that $\mathrm{dist}(u,v) = k$, $0 \leqslant k \leqslant D$, the numbers

$$s_{i1}^k(u,v) := |\Gamma_i^+(u) \cap \Gamma_1^+(v)|, \tag{1}$$

for each i such that $0 \leqslant i \leqslant k+1$, do not depend on the chosen vertices u and v, but only on their distance k. In this case, we just write s_{i1}^k and refer to them as the *intersection numbers*. (According to Damerell's notation [9], s_{i1}^k would be just s_{ik}.) As happens in the undirected case, the notion of distance-regularity in digraphs can be seen as a generalization of the concept of distance-transitivity. The concept of distance-transitive digraphs was introduced by Lam [19] as a natural generalization of distance-transitive graphs. A digraph Γ is said to be *distance-transitive* if for all vertices u, v, x, y such that $\mathrm{dist}(u,v) = \mathrm{dist}(x,y)$ there is an automorphism π of Γ such that $\pi(u) = x$ and $\pi(v) = y$. That paper provided some examples of distance-transitive digraphs with girth g and diameter $D = g-1$, as the directed cycle and the Paley tournament, and presented a method to construct more distance-transitive digraphs, with $D = g$, starting from a distance-transitive digraph with $D = g-1$. Damerell [9] proved that every distance-transitive digraph Γ with girth g, say, is *stable*; that is, $\mathrm{dist}(u,v) + \mathrm{dist}(v,u) = g$ for any pair of vertices $u, v \in V(\Gamma)$ at distance $0 < \mathrm{dist}(u,v) < g$. (Thus, a stable digraph of girth two can be identified with the graph obtained by replacing each pair of opposite arcs, or 'digon', by an undirected edge.) Consequently, every distance-transitive digraph has diameter $D = g$ ('*long*' *distance-transitive digraph*) or $D = g-1$ ('*short distance-transitive digraph*), provided that $g \geqslant 3$. Moreover, Damerell showed that the classification of these digraphs can be reduced to the case $g = D+1$ (i.e., the short digraphs) since every long digraph is obtained from a short digraph, with the same girth, by the construction described by Lam. Using these results, Bannai et al. [1] proved that the distance-transitive digraphs given by Lam were the only ones with odd girth.

The above results were extended by Damerell to the case of distance-regular digraphs. He also gave the classification in 'short' and 'long' digraphs, and the structure of the long digraphs. Taking into account that the case $g = 2$ corresponds to the distance-regular graphs, the study of the distance-regular digraphs is reduced to the case $g \geqslant 3$ (and $g = D+1$). In particular, there is a correspondence between the distance-regular digraphs with girth $g = 3$ and diameter $D = 2$ and the so-called *skew Hadamard matrices*. For $g = 4, 5, 6$, there are some works where the intersection numbers are computed by using the smallest number of defining parameters (see e.g. [12,21]). Finally, Douglas and Nomura [10] showed that the girth g of a distance-regular digraph with degree $\Delta > 1$ and diameter D always satisfies $D \leqslant g \leqslant 8$.

If we change $\Gamma_1^+(v)$ to $\Gamma_1^-(v)$ in the definition of distance-regularity, we get some new parameters p_{i1}^k, called again the intersection numbers. But now the stability property does not necessarily hold, and a class of digraphs with less structure appears. We focus our attention on such digraphs, here referred to as 'weakly distance-regular', and study some of their properties, which are closely related to the properties enjoyed by the distance-regular digraphs. In fact, the diameter 2 weakly distance-regular digraphs are the same as the 'directed strongly regular graphs' introduced by Duval [11], which have recently been studied by a number of authors and found several constructions, see [5]. In our more general context, and besides several characterizations given in the next section, it is shown that every (standard) distance-regular digraph is also weakly distance-regular, so justifying the name. In this case, the relationship between the defining parameters, s_{i1}^k and p_{i1}^k, shows to be very simple. Subsequently, we devote Section 4 to the study of the spectra of weakly distance-regular digraphs, providing formulas for computing the eigenvalue multiplicities from the entries of some different 'small' matrices. Finally, some constructions and specific families of weakly distance-regular digraphs, which are interesting for their applications in network theory, are analyzed.

Since most of our study is of an algebraic nature, we next recall some background from algebraic graph theory and matrix theory. The adjacency matrix $\boldsymbol{A} = (a_{uv})$ of a digraph Γ (we assume[1] that Γ contains neither loops nor multiple edges) is the $N \times N$ matrix indexed by the vertices of Γ, with entries $a_{uv} = 1$ if u is adjacent to v, and $a_{uv} = 0$ otherwise. The spectrum of the digraph Γ, denoted by $\operatorname{sp}\Gamma$, consists of the eigenvalues of $\boldsymbol{A}$, which might be not real since $\boldsymbol{A}$ is not symmetric, together with their (algebraic) multiplicities:

$$\operatorname{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\}.$$

The distance-k matrix $\boldsymbol{A}_k$ of a digraph Γ with diameter D, where $0 \leqslant k \leqslant D$, is defined by

$$(\boldsymbol{A}_k)_{uv} := \begin{cases} 1 & \text{if } \operatorname{dist}(u,v) = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In particular, $\boldsymbol{A}_0 = \boldsymbol{I}$ and $\boldsymbol{A}_1 = \boldsymbol{A}$. As we will see, these matrices play a key role in the study of distance-regularity. For instance, it is known that a digraph Γ is distance-regular (in the sense of Damerell) if and only if the set of matrices $\{\boldsymbol{A}_k\}_{k=0}^{D}$ satisfies the axioms of a commutative association scheme; see [26]. (The readers who are not familiar with association schemes can consult, for instance, Godsil's textbook [17].) This important result is going to be used later.

[1] If a weakly distance-regular digraph Γ contains a loop, then each vertex has the same number of loops. Moreover, if Γ contains a multiple edge, then all edges of Γ are multiple (with equal "multiplicity"). Hence, the deletion of all such repetitions and loops provides another weakly distance-regular digraph.

Some other basic results from algebraic graph theory are the following:

1. By the Perron–Frobenius theorem, the maximum eigenvalue λ_0 of Γ is simple and has a positive eigenvector $\boldsymbol{v}$. In particular, if Γ is Δ-regular, then $\boldsymbol{v}=\boldsymbol{j}$, where $\boldsymbol{j}$ denotes the all-1 vector, and $\lambda_0=\Delta$.
2. The number of walks of length $l\geqslant 0$ from vertex u to vertex v is equal to $a_{uv}^{(l)} := (\boldsymbol{A}^l)_{uv}$.
3. The *adjacency algebra* of Γ (also called *Bose-Mesner algebra* when it is closed under the Hadamard—or componentwise—product) is defined by

   $$\mathscr{A}(\Gamma) := \{p(\boldsymbol{A}) : p\in\mathbb{C}[x]\},$$

   where $\boldsymbol{A}$ is the adjacency matrix of Γ. The dimension of $\mathscr{A}(\Gamma)$, as a $\mathbb{C}$-vector space, equals the degree of the minimum polynomial m_Γ of Γ, and it is at least $D+1$ since the powers $\boldsymbol{I},\boldsymbol{A},\dots,\boldsymbol{A}^D$ are linearly independent.
4. A digraph Γ is (strongly) connected and regular if and only if there is a polynomial $p\in\mathbb{Q}[x]$ such that $p(\boldsymbol{A})=\boldsymbol{J}$, where $\boldsymbol{J}$ denotes the all-1 matrix (see [6, Theorem 5.3.1]). The polynomial H_Γ of least degree satisfying this property is called the *Hoffman polynomial* of Γ and it is given by

   $$H_\Gamma := \frac{N}{S(\Delta)}S, \tag{3}$$

   where $m_\Gamma := (x-\Delta)S\in\mathbb{Z}[x]$ is the minimum polynomial of Γ.

We end this introductory section by recalling some results from matrix theory (see e.g. [20]). Given a complex matrix $\boldsymbol{A}$, we denote by $\boldsymbol{A}^*$ the transpose of its conjugate. Thus, $\boldsymbol{A}$ is *hermitian* if $\boldsymbol{A}^*=\boldsymbol{A}$, and it is *unitary* when $\boldsymbol{A}\boldsymbol{A}^*=\boldsymbol{I}$. Moreover, $\boldsymbol{A}$ is said to be *normal* if $\boldsymbol{A}\boldsymbol{A}^*=\boldsymbol{A}^*\boldsymbol{A}$. The following result summarizes several characterizations of normal matrices.

Theorem 1.1. *Let $\boldsymbol{A}$ be an $n\times n$ complex matrix with eigenvalues $\theta_1,\theta_2,\dots,\theta_n$. Then $\boldsymbol{A}$ is normal if and only if any of the following assertions hold*:

(a) *$\boldsymbol{A}$ diagonalizes by means of a unitary matrix*; *that is, $\boldsymbol{U}^*\boldsymbol{A}\boldsymbol{U}=\boldsymbol{D}$ for some matrix $\boldsymbol{U}$ such that $\boldsymbol{U}\boldsymbol{U}^*=\boldsymbol{I}$, and $\boldsymbol{D}:=\operatorname{diag}(\theta_1,\theta_2,\dots,\theta_n)$.*
(b) *$\boldsymbol{A}^*=p(\boldsymbol{A})$ for some polynomial $p\in\mathbb{C}[x]$.*
(c) $\operatorname{tr}(\boldsymbol{A}\boldsymbol{A}^*)=\sum_{i=1}^n|\theta_i|^2$.

Note that, from (a), the eigenvectors of a normal $n\times n$ square matrix constitute an orthogonal basis of the vector space $\mathbb{C}^n$, with the Hermitian scalar product $\langle\boldsymbol{u},\boldsymbol{v}\rangle_* := \boldsymbol{u}^\top\bar{\boldsymbol{v}}$. Let us now assume that $\boldsymbol{A}$ has distinct eigenvalues $\lambda_0,\lambda_1,\dots,\lambda_d$. For each λ_i, let $\boldsymbol{U}_i$ be the matrix whose columns form an orthonormal basis of the eigenspace $\mathscr{E}_i := \operatorname{Ker}(\boldsymbol{A}-\lambda_i\boldsymbol{I})$. Then the orthogonal projection onto $\mathscr{E}_i$ is represented by the matrix $\boldsymbol{E}_i := \boldsymbol{U}_i\boldsymbol{U}_i^*$ or, alternatively,

$$\boldsymbol{E}_i = \frac{1}{\phi_i}\prod_{j\neq i}(\boldsymbol{A}-\lambda_j\boldsymbol{I}) \quad (0\leqslant i\leqslant d), \tag{4}$$

where $\phi_i \coloneqq \prod_{j=0(j\neq i)}^{d}(\lambda_i-\lambda_j)$. These (hermitian) matrices are called the (*principal*) *idempotents* of $\boldsymbol{A}$, and satisfy the following properties: $\boldsymbol{E}_i\boldsymbol{E}_j=\delta_{ij}\boldsymbol{E}_i$; $\boldsymbol{A}\boldsymbol{E}_i=\lambda_i\boldsymbol{E}_i$; $\operatorname{sp}\boldsymbol{E}_i=\{1^{m(\lambda_i)},0^{n-m(\lambda_i)}\}$; and, for every rational function f defined at each λ_i, $0\leqslant i\leqslant d$,

$$f(\boldsymbol{A})=\sum_{i=0}^{d} f(\lambda_i)\boldsymbol{E}_i. \tag{5}$$

In particular, when $f=x$, the above equation becomes the so-called *spectral decomposition theorem*: $\boldsymbol{A}=\sum_{i=0}^{d}\lambda_i\boldsymbol{E}_i$.

2. Weak distance-regularity

We begin this section by formally introducing the concept of weakly distance-regular digraph, through the invariance of the number of walks between vertices at a given distance. (This approach has already been used to characterize distance-regular graphs, see e.g. [25].) Afterwards, and as already mentioned, we will show that this concept is equivalent to change $\Gamma_1^+(v)$ to $\Gamma_1^-(v)$ in the definition of a distance-regular digraph.

Definition 2.1. A digraph Γ of diameter D is *weakly distance-regular* if, for each non-negative integer $l\leqslant D$, the number $a_{uv}^{(l)}$ of walks of length l from vertex u to vertex v only depends on their distance $\operatorname{dist}(u,v)=k$, for any $l=0,1,\ldots,D$. In this case we write $a_{uv}^{(l)}=a_k^l$ for some constants a_k^l, $0\leqslant k,l\leqslant D$.

Of course $a_k^l=0$ for any $k>l$, since there is no walk of length l between vertices u,v at distance $\operatorname{dist}(u,v)>l$. Moreover, if the digraph is geodetic (that is, the shortest path between any two vertices is unique), then $a_k^k=1$ for any $0\leqslant k\leqslant D$.

From the definition of Γ as a weakly distance-regular digraph, it follows that each matrix power $\boldsymbol{A}^l$, $0\leqslant l\leqslant D$, can be expressed as a linear combination of the distance matrices $\boldsymbol{A}_0,\boldsymbol{A}_1,\ldots,\boldsymbol{A}_D$ which, as we will see next, belong to the adjacency algebra of Γ. The following theorem provides some characterizations of weakly distance-regular digraphs which are analogous to those applying for distance-regular graphs (for a survey of the latter, see e.g. [15]).

Theorem 2.2. *Let Γ be a connected digraph with diameter D and distance matrices $\{\boldsymbol{A}_k\}_{k=0}^{D}$. Then, the following are equivalent*:

(w) *Γ is a weakly distance-regular digraph.*
(a) *The distance matrix $\boldsymbol{A}_k$ is a polynomial of degree k in the adjacency matrix $\boldsymbol{A}$; that is, $\boldsymbol{A}_k=p_k(\boldsymbol{A})$, for each $k=0,1,\ldots,D$, where $p_k\in\mathbb{Q}[x]$.*
(b) *The set of distance matrices $\{\boldsymbol{A}_k\}_{k=0}^{D}$ is a basis of the adjacency algebra $\mathscr{A}(\Gamma)$.*

(c) *For any two vertices $u, v \in V(\Gamma)$ at distance* $\mathrm{dist}(u,v) = k$, *the numbers*

$$p_{ij}^k(u,v) := |\Gamma_i^+(u) \cap \Gamma_j^-(v)| \tag{6}$$

do not depend on the vertices u and v, but only on their distance k; in which case they are denoted by p_{ij}^k (note that $p_{ij}^k = 0$ when $k > i + j$).

Proof. (w) $\Leftrightarrow$ (a): If Γ is weakly distance-regular, the number of walks of length $l \leqslant D$ from vertices u, v at distance k is $a_{uv}^{(l)} = a_k^l$ (a constant). Hence,

$$\boldsymbol{A}^l = a_0^l \boldsymbol{I} + a_1^l \boldsymbol{A} + a_2^l \boldsymbol{A}_2 + \cdots + a_l^l \boldsymbol{A}_l \quad (0 \leqslant l \leqslant D), \tag{7}$$

where, necessarily, $a_l^l \neq 0$ and, as already mentioned, $a_k^l = 0$ for any $k > l$. In matrix form

$$\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \boldsymbol{A}^2 \\ \cdot \\ \cdot \\ \boldsymbol{A}^D \end{pmatrix} = \begin{pmatrix} a_0^0 & & & & & \\ a_0^1 & a_1^1 & & & & \\ a_0^2 & a_1^2 & a_2^2 & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_0^D & a_1^D & \cdot & \cdot & \cdot & a_D^D \end{pmatrix} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \boldsymbol{A}_2 \\ \cdot \\ \cdot \\ \boldsymbol{A}_D \end{pmatrix}, \tag{8}$$

where $\boldsymbol{C} := (a_k^l)$ is a lower triangular matrix. Since $a_l^l > 0$ for any $0 \leqslant l \leqslant D$, the matrix $\boldsymbol{C}$ is non-singular and its inverse $\boldsymbol{C}^{-1}$ is also a lower triangular matrix. Hence $\boldsymbol{A}_k$ is a polynomial of degree k in $\boldsymbol{A}$:

$$\boldsymbol{A}_k = p_k(\boldsymbol{A}) = \alpha_0^k \boldsymbol{I} + \alpha_1^k \boldsymbol{A} + \alpha_2^k \boldsymbol{A}^2 + \ldots + \alpha_k^k \boldsymbol{A}^k \quad (0 \leqslant k \leqslant D), \tag{9}$$

where $\alpha_k^k \neq 0$. Conversely, let us assume that there are constants α_l^k, with $\alpha_k^k \neq 0$, satisfying (9). This implies that Eqs. (7) and (8) also hold with $\boldsymbol{C} = (a_k^l)$ being the inverse matrix of $\boldsymbol{C}^{-1} = (\alpha_l^k)$ and, hence, the number of walks of length $l \leqslant D$ from one vertex to another only depends on their distance.

(a) $\Rightarrow$ (b): We know that $\{\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}_D\}$ constitutes a set of linearly independent matrices of the adjacency algebra $\mathscr{A}(\Gamma)$. Moreover, since $\boldsymbol{A}_k = p_k(\boldsymbol{A})$ and $\sum_{k=0}^D \boldsymbol{A}_k = \boldsymbol{J}$, it follows that $H(\boldsymbol{A}) = \boldsymbol{J}$, where $H := \sum_{k=0}^D p_k$. As a consequence, Γ is a regular digraph of degree Δ, say. Furthermore, since the minimum polynomial m_Γ of Γ divides $(x - \Delta)H$ we have $D + 1 \leqslant \dim \mathscr{A}(\Gamma) = \mathrm{dgr}\, m_\Gamma \leqslant D + 1$. Hence, the dimension of $\mathscr{A}(\Gamma)$ attains the minimum value allowed by the diameter, $\dim \mathscr{A}(\Gamma) = D + 1$, and both $\{\boldsymbol{A}^l\}_{l=0}^D$ and $\{\boldsymbol{A}_k\}_{k=0}^D$ are bases of $\mathscr{A}(\Gamma)$.

(b) $\Rightarrow$ (c): Let u and v be two vertices of the digraph Γ, such that $\mathrm{dist}(u,v) = k$. Notice that the number $p_{ij}^k(u,v)$, representing the number of vertices at distance i from u and at distance j to v, coincides with the (u,v)-entry of the matrix $\boldsymbol{A}_i \boldsymbol{A}_j$ which, because of (b), is a linear combination of the basis $\{A_k\}_{k=0}^{D}$, say,

$$A_i A_j = \sum_{k=0}^{D} \gamma_{ij}^{k} A_k. \tag{10}$$

Consequently, $p_{ij}^{k}(u,v) = \gamma_{ij}^{k} = p_{ij}^{k}$ for any two vertices u, v at distance k.

(c) $\Rightarrow$ (a): If (10) holds for $j = 1$:

$$A_i A_1 = \sum_{k=0}^{D} p_{i1}^{k} A_k \quad (0 \leqslant i \leqslant D), \tag{11}$$

we can use an inductive argument, starting from $A_0 = I$ and $A_1 = A$, to deduce that the distance matrix A_k is indeed a polynomial of degree k in the adjacency matrix A. $\square$

From now on, the polynomials p_k such that $A_k = p_k(A)$, $0 \leqslant k \leqslant D$, will be referred to as the *distance polynomials* of Γ. Notice that, from (b), any matrix power A^l is a linear combination of the distance matrices and, consequently, the number $a_{uv}^{(l)}$ of walks of *any* fixed length $l \geqslant 0$ between vertices u, v of Γ only depends on their distance. Other interesting consequences of the above theorem are the following.

Corollary 2.3. *Let Γ be a weakly distance-regular digraph on N vertices, with diameter D, adjacency matrix A, and distance polynomials $\{p_k\}_{k=0}^{D}$. Then,*

(i) *The Hoffman polynomial of Γ is $H_\Gamma = \sum_{k=0}^{D} p_k$. In particular, Γ is a regular digraph of degree Δ, where $\sum_{k=0}^{D} p_k(\Delta) = N$. Moreover, the minimum polynomial of Γ is $m_\Gamma = (x - \Delta)\frac{1}{\alpha_D^D} H_\Gamma$, where α_D^D is the leading coefficient of p_D.*

(ii) *The number n_k of vertices at distance k from any given vertex is equal to $p_k(\Delta)$, for each $k = 0, 1, \ldots, D$.*

(iii) *The spectrum of Γ is uniquely determined by the entries of the $(D+1) \times (D+1)$* matrix $C = (a_k^l)$.

Proof. (i)–(ii): From the proof of Theorem 2.2(b), we see that $H_\Gamma = H = \sum_{k=0}^{D} p_k$, Γ is a Δ-regular digraph, and the minimum polynomial m_Γ is as claimed. Moreover, we have that $j = (1, 1, \ldots, 1)^\top$ is an eigenvector of A corresponding to the eigenvalue Δ. Consequently, $A_k j = p_k(A) j = p_k(\Delta) j$, which implies that $H_\Gamma(\Delta) = \sum_{k=0}^{D} p_k(\Delta) = N$, in concordance with (3).

(iii) This follows from the fact that the characteristic polynomial of Γ is uniquely determined by the minimum polynomial m_Γ and the spectral invariants $\operatorname{tr} A^l = N a_0^l$, $0 \leqslant l \leqslant D$ (see Section 3 for more details). $\square$

As we will see, in the case of weakly distance-regularity, the average of the intersection numbers (1) also play a role. In a given digraph with diameter D, let N_k,

$0 \leqslant k \leqslant D$, denote the number of (ordered) vertex pairs u, v such that $\mathrm{dist}(u,v) = k$; that is, $N_k := \sum_{u \in V} n_k(u)$. Thus, $N_0 = N$ and $N_1 = |E|$. Then the *mean intersection numbers* $\bar{s}_{i1}^k$ are defined by

$$\bar{s}_{i1}^k := \frac{1}{N_k} \sum_{\mathrm{dist}(u,v)=k} s_{i1}^k(u,v) \tag{12}$$

and, more generally, we define $\bar{s}_{ij}^k$ as the average of the numbers $s_{ij}^k(u,v) := |\Gamma_i^+(u) \cap \Gamma_j^+(v)|$.

To see what is the significance of these parameters, let us consider, for a given digraph Γ with adjacency matrix $\boldsymbol{A}$, the following scalar product in $\mathbb{C}[x]$:

$$\langle f, g \rangle_\Gamma := \frac{1}{N} \mathrm{tr}(f(\boldsymbol{A}) g(\boldsymbol{A})^*). \tag{13}$$

In fact, notice that this product is well defined in the quotient ring $\mathbb{C}[x]/\mathscr{I}$, where $\mathscr{I} := (m_\Gamma)$ is the ideal generated by the minimum polynomial of Γ. This is due to the additive property of the trace and $\mathrm{tr}(m_\Gamma(\boldsymbol{A})) = \mathrm{tr}\,\boldsymbol{O} = 0$.

Then, if Γ is a weakly distance-regular digraph, its distance polynomials are orthogonal with respect to this product. Indeed,

$$\langle p_k, p_l \rangle_\Gamma = \frac{1}{N} \mathrm{tr}(\boldsymbol{A}_k \boldsymbol{A}_l^\top) = \frac{1}{N} \sum_{u \in V} |\Gamma_k^+(u) \cap \Gamma_l^+(u)| = 0 \quad (k \neq l) \tag{14}$$

and

$$\|p_k\|_\Gamma^2 = \frac{1}{N} \mathrm{tr}(\boldsymbol{A}_k \boldsymbol{A}_k^\top) = n_k = p_k(\Delta). \tag{15}$$

Consequently, the set $\{p_k\}_{k=0}^D$ is a basis of the quotient ring $\mathbb{C}[x]/\mathscr{I}$, which is isomorphic to $\mathscr{A}(\Gamma)$. The representation of $p_i x$ in terms of such a basis must be of the form

$$p_i x = \sum_{k=0}^{D} \gamma_i^k p_k \quad (0 \leqslant i \leqslant D),$$

where γ_i^k is the corresponding Fourier coefficient. Comparing with (11), we conclude that γ_i^k must be equal to the intersection number p_{i1}^k (a real number). Hence,

$$\gamma_i^k = \overline{\gamma_i^k} = \frac{\langle p_i x, p_k \rangle_\Gamma}{\|p_k\|_\Gamma^2} = \frac{\langle p_k, p_i x \rangle_\Gamma}{\|p_k\|_\Gamma^2} \quad (0 \leqslant i \leqslant D). \tag{16}$$

In the following result, we show that these Fourier coefficients can also be computed from the average intersection numbers $\bar{s}_{ij}^k$.

Proposition 2.4. *The coefficients of the above recurrence satisfied by the distance polynomials of a weakly distance-regular digraph are*

$$\gamma_i^k = p_{i1}^k = \frac{N_i}{N_k} \bar{s}_{k1}^i.$$

Proof. If we compute γ_i^k by using the first expression in (16), we get

$$\begin{aligned}\gamma_i^k &= \frac{\langle p_i x, p_k \rangle_\Gamma}{\|p_k\|_\Gamma^2} = \frac{\mathrm{tr}(\boldsymbol{A}_i \boldsymbol{A} \boldsymbol{A}_k^\top)}{\mathrm{tr}(\boldsymbol{A}_k \boldsymbol{A}_k^\top)} = \frac{\sum_{u \in V} \sum_{v \in V} (\boldsymbol{A}_i)_{uv} (\boldsymbol{A}\boldsymbol{A}_k^\top)_{vu}}{\sum_{u \in V} \sum_{v \in V} (\boldsymbol{A}_k)_{uv} (\boldsymbol{A}_k)_{uv}} \\ &= \frac{1}{N_k} \sum_{u \in V} \sum_{v \in \Gamma_i^+(u)} (\boldsymbol{A}\boldsymbol{A}_k^\top)_{vu} = \frac{1}{N_k} \sum_{u \in V} \sum_{v \in \Gamma_i^+(u)} (\boldsymbol{A}_k \boldsymbol{A}^\top)_{uv} \\ &= \frac{1}{N_k} \sum_{u \in V} \sum_{v \in \Gamma_i^+(u)} s_{k1}^i(u,v) = \frac{N_i}{N_k} \bar{s}_{k1}^i.\end{aligned}$$

This completes the proof, as we already know that $\gamma_i^k = p_{i1}^k$ (in fact, this is what we get if we compute γ_i^k by using the second expression in (16)). $\square$

In general, as a consequence of Theorem 2.2 and $\langle p_k, p_i p_j \rangle_\Gamma = \langle p_i p_j, p_k \rangle_\Gamma$, we reach the following conclusion:

Corollary 2.5. *A distance-regular digraph Γ with intersection numbers s_{ij}^k is also a weakly distance-regular digraph with intersection parameters p_{ij}^k satisfying*

$$N_k p_{ij}^k = N_i s_{kj}^i. \tag{17}$$

Proof. Since Γ is distance-regular, its adjacency algebra is a (P-polynomial) association scheme (that is, each distance matrix $\boldsymbol{A}_k$, $0 \leqslant k \leqslant D$, is a polynomial p_k of degree k in $\boldsymbol{A}$) and so Theorem 2.2(a) applies. This assures that the numbers p_{ij}^k exist and the proof goes as before. (In fact, notice that (17) just counts in two ways the number of ordered triples (x,y,z) such that $\mathrm{dist}(x,y)=k$, $\mathrm{dist}(x,z)=i$, and $\mathrm{dist}(z,y)=j$.) $\square$

Thus, any distance-regular digraph Γ satisfies all the above properties of weakly distance-regular digraphs. Moreover, since Γ is stable and has girth $g \in \{2, D, D+1\}$, see [9], we have

$$\boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{A}\boldsymbol{A}_{g-1} = \boldsymbol{A} p_{g-1}(\boldsymbol{A}) = \boldsymbol{A}_{g-1}\boldsymbol{A} = \boldsymbol{A}^\top \boldsymbol{A}$$

(since $1 \leqslant g-1 \leqslant D$) and $\boldsymbol{A}$ is a normal matrix.

As a consequence of the above, we now have the following characterization result, showing that the (standard) distance-regular digraphs are precisely those weakly distance-regular digraphs which are stable or have a normal adjacency matrix.

Proposition 2.6. *Let Γ be a weakly distance-regular digraph with adjacency matrix $\boldsymbol{A}$. Then Γ is distance-regular if and only if any of the following conditions hold*:

(a) *Γ is stable.*
(b) *$\boldsymbol{A}$ is normal.*

Proof. We only need to prove that each of the conditions is sufficient.

(a) Let Γ be a weakly distance-regular digraph with diameter D. From the proof of Theorem 2.2, we already know that $\boldsymbol{A}_i\boldsymbol{A}_j = \boldsymbol{A}_j\boldsymbol{A}_i$ is a linear combination of the basis $\{\boldsymbol{A}_k\}_{k=0}^{D}$, see (10). On the other hand, since Γ is stable, we have that $\boldsymbol{A}_k^{\top} = \boldsymbol{A}_{g-k}$, $0<k<g$, where g is the girth of Γ. If $g=2$, then Γ is a symmetric digraph since $\boldsymbol{A}^{\top} = \boldsymbol{A}$ and, consequently, $\boldsymbol{A}_k^{\top} = \boldsymbol{A}_k$ for each $k=0,1,\dots,D$. If $g\geqslant 3$, then $g=D$ or $g=D+1$ (see [9, Theorem 2]), which implies that $\boldsymbol{A}_k^{\top} = \boldsymbol{A}_{D-k}$ $(0<k<D)$ and $\boldsymbol{A}_D^{\top} = \boldsymbol{A}_D$, if $g=D$, and $\boldsymbol{A}_k^{\top} = \boldsymbol{A}_{D+1-k}$, if $g=D+1$ and $0<k\leqslant D$. Thus, the adjacency algebra $\mathscr{A}(\Gamma)$ is closed under the "transposition operation". Hence, the set of matrices $\{\boldsymbol{A}_k\}_{k=0}^{D}$ satisfies the axioms of a (commutative) association scheme and Γ is a distance-regular digraph.

(b) If $\boldsymbol{A}$ is normal, then by Theorem 1.1(b) there exists a polynomial $p\in\mathbb{C}[x]$ such that $\boldsymbol{A}^{\top} = p(\boldsymbol{A})$. Thus, $\boldsymbol{A}_i\boldsymbol{A}^{\top}\in\mathscr{A}(\Gamma)$ and, hence, it admits a representation in terms of the basis $\{\boldsymbol{A}_k\}_{k=0}^{D}$, which must be of the form

$$\boldsymbol{A}_i\boldsymbol{A}^{\top} = \sum_{k=0}^{D} \sigma_i^k \boldsymbol{A}_k \quad (0\leqslant i\leqslant D)$$

since $(\boldsymbol{A}_i\boldsymbol{A}^{\top})_{uv} = |\Gamma_i^+(u)\cap\Gamma^+(v)|$. Therefore, as the Fourier coefficients σ_i^k only depend on k (and i), they coincide with the intersection numbers s_{i1}^k and Γ is distance-regular. $\square$

From this result, we can see now that the scalar product associated to a distance-regular digraph has a simple expression in terms of its spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \dots, \lambda_d^{m(\lambda_d)}\}$. Indeed, using Theorem 1.1 and the properties of the idempotents, (13) becomes

$$\langle f,g\rangle_\Gamma = \frac{1}{N}\sum_{i=0}^{d} f(\lambda_i)\overline{g(\lambda_i)}\operatorname{tr}(\boldsymbol{E}_i\boldsymbol{E}_i^*) = \frac{1}{N}\sum_{i=0}^{d} m(\lambda_i)f(\lambda_i)\overline{g(\lambda_i)} \tag{18}$$

which, except for the conjugation, is like the scalar product used for distance-regular graphs (see [15]). In the next section, we will derive the corresponding expression for a weak distance-regular digraph.

3. The spectrum and the condensed matrices

In this section we study how to compute the spectrum of a weakly distance-regular digraph in terms of its defining parameters. We will see that the whole spectrum of such a digraph can be retrieved from the information given by some specific matrices characterizing it, such as the recurrence matrix or the multiplicity matrix, which have size much more smaller than the adjacency matrix. Thus, these matrices are here generically referred to as the 'condensed matrices'.

3.1. The multiplicities

First recall that, by Corollary 2.3(i), the distinct eigenvalues of a weakly distance-regular digraph Γ with distance polynomials $\{p_k\}_{k=0}^D$ are $\lambda_0 = \Delta$ and the different zeros $\lambda_1, \lambda_2, \ldots, \lambda_d$ $(d \leqslant D)$ of its Hoffman polynomial $H_\Gamma = \sum_{k=0}^D p_k$. Moreover, their respective multiplicities can be computed by solving the system of linear equations

$$\operatorname{tr} \boldsymbol{A}^j = \sum_{i=0}^{d} m(\lambda_i)\lambda_i^j = N a_0^j \quad (0 \leqslant j \leqslant d), \tag{19}$$

where the independent terms $a_0^j = a_{uu}^{(j)}$ can be computed from the coefficients of the distance polynomials (see the proof of Theorem 2.2); and the coefficient matrix is a Vandermonde matrix formed from the distinct values $\lambda_0, \lambda_1, \ldots, \lambda_d$.

However, in our context, a more direct way of computing the multiplicities is to consider the traces of the distance matrices. Indeed, since

$$\operatorname{tr} \boldsymbol{A}_i = \operatorname{tr}(p_i(\boldsymbol{A})) = \sum_{j=0}^{d} m(\lambda_j) p_i(\lambda_j) = \delta_{0i} N \quad (0 \leqslant i \leqslant d) \tag{20}$$

(notice that $\operatorname{tr} \boldsymbol{A}_i = N \langle p_i, p_0 \rangle_\Gamma$) we have the following result:

Proposition 3.1. *Let Γ be a weakly distance-regular digraph with N vertices, distance polynomials $\{p_k\}_{k=0}^D$, and distinct eigenvalues $\lambda_0(=\Delta), \lambda_1, \ldots, \lambda_d$. Let $\boldsymbol{P}_{\text{ev}}$ be the matrix whose (i,j)-element is $p_i(\lambda_j)$, $0 \leqslant i,j \leqslant d$. Then the multiplicities $m(\lambda_i)$ are given by*

$$m(\lambda_i) = N(\boldsymbol{P}_{\text{ev}}^{-1})_{i0} \quad (0 \leqslant i \leqslant d). \tag{21}$$

Proof. Since $\boldsymbol{P}_{\text{ev}}$ is clearly non-singular, (20) yields

$$\begin{pmatrix} m(\lambda_0) \\ m(\lambda_1) \\ \vdots \\ m(\lambda_d) \end{pmatrix} = \begin{pmatrix} p_0(\lambda_0) & p_0(\lambda_1) & \cdots & p_0(\lambda_d) \\ p_1(\lambda_0) & p_1(\lambda_1) & \cdots & p_1(\lambda_d) \\ \vdots & \vdots & \vdots & \vdots \\ p_d(\lambda_0) & p_d(\lambda_1) & \cdots & p_d(\lambda_d) \end{pmatrix}^{-1} \begin{pmatrix} N \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{22}$$

which corresponds to (21). □

In the following subsections, we study other matrices related to Γ, which also contain the information about its multiplicities. We begin with a positive definite matrix which gives an alternative expression for the scalar product (13).

3.2. The multiplicity matrix

Let Γ be a weakly distance-regular digraph with diameter D, and adjacency matrix $\boldsymbol{A}$ with distinct eigenvalues $\lambda_0(=\Delta), \lambda_1, \ldots, \lambda_d$, where $d \leqslant D$. Let $m_0, m_1, \ldots, m_d$

denote the multiplicities of such eigenvalues, as zeros of the minimum polynomial m_Γ. Thus, $m_0 = 1$, $1 \leqslant m_i \leqslant m(\lambda_i)$ for any $1 \leqslant i \leqslant d$, and $\sum_{i=0}^{d} m_i = D + 1$. For every polynomial $f \in \mathbb{C}[x]$, let us define the (row) vector $\boldsymbol{f} = (f_0, f_1, \dots, f_D) \in \mathbb{C}^{D+1}$ as

$$\boldsymbol{f} := \left(f(\lambda_0), f(\lambda_1), f'(\lambda_1), \frac{f''(\lambda_1)}{2!}, \dots, \frac{f^{(m_1-1)}(\lambda_1)}{(m_1-1)!}, \dots, f(\lambda_d), \right.$$
$$\left. f'(\lambda_d), \dots, \frac{f^{(m_d-1)}(\lambda_d)}{(m_d-1)!} \right),$$

where the superscripts stand for the orders of the derivatives. Then we claim that there exists a positive definite matrix $\boldsymbol{M} = (m_{rs})$, $0 \leqslant r, s \leqslant D$, such that the scalar product (13) can be written as

$$\langle f, g \rangle_\Gamma = \frac{1}{N} \boldsymbol{f} \boldsymbol{M} \boldsymbol{g}^* = \frac{1}{N} \sum_{r,s=0}^{D} m_{rs} f_r \overline{g_s}. \tag{23}$$

Moreover, the row (or column) sums of a given submatrix $\boldsymbol{M}_{\mathrm{ev}}$ of $\boldsymbol{M}$ yield the multiplicities of Γ. More precisely let us consider the set of indexes $\mathscr{J} := \{0, \sum_{j=0}^{i-1} m_j : 1 \leqslant i \leqslant d\} \subseteq \mathbb{Z}_{D+1}$, and let $\boldsymbol{M}_{\mathrm{ev}}$ denote the principal submatrix of $\boldsymbol{M}$ whose row and column indexes are the $(d+1)$ elements of $\mathscr{J}$. Then

$$\sum_{j=0}^{d} (\boldsymbol{M}_{\mathrm{ev}})_{ij} = m(\lambda_i) \quad (0 \leqslant i \leqslant d).$$

Since the distance polynomials form a basis, to prove the first statement it is enough to show that there is a matrix $\boldsymbol{M}$ satisfying (14) and (15); that is, $\frac{1}{N} \boldsymbol{p}_k \boldsymbol{M} \boldsymbol{p}_l^* = \delta_{kl} p_k(\Delta)$ or, in matrix form

$$\frac{1}{N} \boldsymbol{P} \boldsymbol{M} \boldsymbol{P}^* = \boldsymbol{D},$$

where $\boldsymbol{P}$ is the matrix with rows $\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_D$ and $\boldsymbol{D} := \mathrm{diag}(p_0(\Delta), p_1(\Delta), \dots, p_D(\Delta))$. Thus, since the vectors $\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_D$ are linearly independent (otherwise, Hermite interpolation at $\lambda_0, \lambda_1, \dots, \lambda_d$ would give a contradiction) the matrix $\boldsymbol{P}$ has an inverse and so

$$\boldsymbol{M} = N \boldsymbol{P}^{-1} \boldsymbol{D} (\boldsymbol{P}^{-1})^* = \boldsymbol{V} \boldsymbol{V}^*, \tag{24}$$

where we have introduced the new (non-singular) matrix $\boldsymbol{V} := \sqrt{N} \boldsymbol{P}^{-1} \boldsymbol{D}^{1/2}$. Hence, $\boldsymbol{M}$ is a definite positive hermitian matrix (note that, for any $\boldsymbol{f} \neq 0$, $\boldsymbol{f} \boldsymbol{M} \boldsymbol{f}^* = ||f||_\Gamma^2 > 0$).

Furthermore,

$$\mathrm{tr}\, \boldsymbol{A}_k = \sum_{i=0}^{d} p_k(\lambda_i) m(\lambda_i) = \boldsymbol{p}_k \boldsymbol{m}^* \quad (0 \leqslant k \leqslant D), \tag{25}$$

where $\boldsymbol{m}$ is a $(D+1)$-vector whose r_ith entry is $m(\lambda_i)$, when $r_i \in \mathscr{J}$ is defined as $r_0 := 0$ and $r_i := m_0 + m_1 + \cdots + m_{i-1}$, $1 \leqslant i \leqslant d$; and $m_r = 0$ if $r \notin \mathscr{J}$. Also,

$$\mathrm{tr}\, \boldsymbol{A}_k = N \langle p_k, p_0 \rangle_\Gamma = \boldsymbol{p}_k \boldsymbol{M} \boldsymbol{p}_0^* \quad (0 \leqslant k \leqslant D). \tag{26}$$

Thus, by (25) and (26),

$$\boldsymbol{p}_k(\boldsymbol{M}\boldsymbol{p}_0^* - \boldsymbol{m}^*) = 0 \quad (0 \leqslant k \leqslant D)$$

and, by the independence of $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_D$, we conclude that $\boldsymbol{M}\boldsymbol{p}_0^* = \boldsymbol{m}^*$, and hence $\boldsymbol{M}_{\mathrm{ev}}\boldsymbol{j} = \boldsymbol{m}_{\mathrm{ev}}^*$ where $\boldsymbol{m}_{\mathrm{ev}} := (m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_d))$. Of course, this gives a way of computing the multiplicities from the entries of $\boldsymbol{M}$ which, in turn, can be computed from the distance polynomials using (24). This gives

$$\begin{aligned}\boldsymbol{m}^* &= \boldsymbol{M}\boldsymbol{p}_0^* = N\boldsymbol{P}^{-1}\boldsymbol{D}(\boldsymbol{P}^{-1})^*\boldsymbol{p}_0^* = N\boldsymbol{P}^{-1}\boldsymbol{D}(\boldsymbol{p}_0\boldsymbol{P}^{-1})^* \\ &= N\boldsymbol{P}^{-1}\boldsymbol{e}_1^*,\end{aligned} \tag{27}$$

where $\boldsymbol{e}_1$ is the 1th coordinate vector. Consequently, $\boldsymbol{m}^*$ is just N times the first column of $\boldsymbol{P}^{-1}$. (A result to be compared with (22) which, with the above notation, reads $\boldsymbol{m}_{\mathrm{ev}}^* = N\boldsymbol{P}_{\mathrm{ev}}^{-1}\boldsymbol{e}_1^*$.) A more direct way of seeing this is to consider again (25). Indeed, as $\operatorname{tr} \boldsymbol{A}_0 = N$ and $\operatorname{tr} \boldsymbol{A}_k = 0$ for any $1 \leqslant k \leqslant D$, Eqs. (25) yield, in matrix form, $\boldsymbol{P}\boldsymbol{m}^* = N\boldsymbol{e}_1^*$, whence (27) follows.

Let us now prove that $m_{0s} = 0$ for any $s \neq 0$ and $m_{00} = 1$. To this end, we now compute the scalar product $\langle Z_{i\varepsilon}, H_\Gamma \rangle_\Gamma$, where $Z_{i\varepsilon} := m_\Gamma/(x-\lambda_i)^\varepsilon$, $1 \leqslant \varepsilon \leqslant m_i$, in two different ways:

$$\langle Z_{i\varepsilon}, H_\Gamma \rangle_\Gamma = \frac{1}{N}\operatorname{tr}(Z_{i\varepsilon}(\boldsymbol{A})H_\Gamma(\boldsymbol{A})^*) = \frac{1}{N}\operatorname{tr}(Z_{i\varepsilon}H_\Gamma(\boldsymbol{A})) = \begin{cases} 0 & \text{if } i \neq 0, \\ Z_{01}(\Delta) & \text{if } i = 0 \end{cases} \tag{28}$$

since $H_\Gamma = \frac{N}{Z_{01}(\Delta)}Z_{01}$ and hence $Z_{i\varepsilon}H_\Gamma$ is a multiple of m_Γ for any $i \neq 0$. Also, from the vectors $\boldsymbol{Z}_{i\varepsilon} = (0, 0, \ldots, 0, Z_{i\varepsilon}^{(m_i-\varepsilon)}(\lambda_i), \ldots, Z_{i\varepsilon}^{(m_i-1)}(\lambda_i), 0, \ldots, 0)$ and $\boldsymbol{H}_\Gamma = (N, 0, 0, \ldots, 0)$ (derived from the respective polynomials), we have

$$\langle Z_{i\varepsilon}, H_\Gamma \rangle_\Gamma = \frac{1}{N}\sum_{r=r_i+m_i-\varepsilon}^{r_i+m_i-1} m_{r0}(Z_{i\varepsilon})_r(H_\Gamma)_0 = \sum_{r=r_i+m_i-\varepsilon}^{r_i+m_i-1} m_{r0}Z_{i\varepsilon}^{(r-r_i)}(\lambda_i). \tag{29}$$

Thus, equating (28) and (29) for $i = 0$ (whence $r_0 = 0$ and $\varepsilon = 1$) we get $m_{00} = 1$. Similarly, for $i \neq 0$ and $\varepsilon = 1$, we obtain $m_{r0}Z_{i\varepsilon}^{(m_i-1)}(\lambda_i) = 0$ where $r = r_i + m_i - 1$. Hence, for such a value of r, we must have $m_{r0} = 0$, as $Z_{i\varepsilon}^{(m_i-1)}(\lambda_i) \neq 0$. Recursively, for the same i and $\varepsilon = 2, \ldots, m_i - 1$ we obtain $m_{r0} = 0$ for $r = r_i + m_i - \varepsilon$.

As a consequence of the above, the scalar product (23) can be rewritten in a slightly more simplified form:

$$\langle f, g \rangle_\Gamma = \frac{1}{N}\left(f_0\overline{g_0} + \sum_{r,s=1}^{D} m_{rs}f_r\overline{g_s} \right). \tag{30}$$

3.3. The recurrence matrix

First, let us consider the $(D+1) \times (D+1)$ matrices $\boldsymbol{B}_i$, $0 \leqslant i \leqslant D$, whose entries are the intersection numbers of Γ ; that is, $(\boldsymbol{B}_i)_{jk} = (p_{ij}^k)$. It is known that the character algebra, generated by the matrices $\boldsymbol{B}_0, \boldsymbol{B}_1, \dots, \boldsymbol{B}_D$, is isomorphic to the adjacency algebra $\mathscr{A}(\Gamma)$ (this is a generalization of a result from [4]). Since such an isomorphism assigns the matrix $\boldsymbol{B} := \boldsymbol{B}_1$ to the matrix $\boldsymbol{A} = \boldsymbol{A}_1$, it turns out that $\boldsymbol{B}_k = p_k(\boldsymbol{B})$, $0 \leqslant k \leqslant D$, and, consequently, it is enough to compute $\boldsymbol{B}_1$, called the *intersection* or *recurrence matrix*. In particular, $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same minimum polynomial and, consequently, the same distinct eigenvalues. As we have already seen, the entries of $\boldsymbol{B} = (p_{1j}^k)$ give the following recurrence satisfied by the distance polynomials:

$$xp_j = \sum_{k=0}^{D} p_{1j}^k p_k \quad (0 \leqslant j \leqslant D), \tag{31}$$

where $p_0 = 1$ and $p_1 = x$. (Remember that polynomial equalities must be understood in the quotient ring $\mathbb{C}[x]/\mathscr{I}$.) Since the entries of $\boldsymbol{B}$ correspond to the cardinalities $p_{1j}^k = |\Gamma_1^+(u) \cap \Gamma_j^-(v)|$, provided that $\mathrm{dist}(u,v) = k$, and Γ is Δ-regular, we infer that $\sum_{j=0}^{D} p_{1j}^k = \Delta$, for any $k = 0, 1, \dots, D$; that is, $\boldsymbol{j}$ is a left eigenvector of B with eigenvalue Δ. In particular, the last row of $\boldsymbol{B}$ is uniquely determined for the other D. In matrix form, the $D+1$ equations in (31) correspond to

$$\boldsymbol{B}\boldsymbol{p}(x) = x\boldsymbol{p}(x), \tag{32}$$

where $\boldsymbol{p}(x) := (p_0(x), p_1(x), \dots, p_D(x))^\top$. Note that, for a given value of x and starting from $p_0(x) = 1$ and $p_1(x) = x$, the jth equation, $2 \leqslant j \leqslant D-1$, allows us to compute the value of $p_{j+1}(x)$ from those of $p_k(x)$, $k = 0, 1, \dots, j$, through the formula

$$\sum_{k=0}^{j-1} p_{1j}^k p_k(x) + (p_{1j}^j - x)p_j(x) + p_{1j}^{j+1} p_{j+1}(x) = 0, \tag{33}$$

whereas the last equation $(j = D)$, which we write in the form

$$p_{D+1}(x) := (x - p_{1D}^D)p_D(x) - \sum_{k=0}^{D-1} p_{1D}^k p_k(x) = 0, \tag{34}$$

yields a condition for $\boldsymbol{p}(\lambda)$ being a (right) eigenvector of $\boldsymbol{B}$ with eigenvalue λ, namely that $p_{D+1}(\lambda) = 0$. (Notice again that $p_{D+1} = 0$ must be understood in the quotient ring $\mathbb{C}[x]/\mathscr{I}$.) From this, we infer that the characteristic polynomial of $\boldsymbol{B}$, which coincides with the minimum polynomial of $\boldsymbol{B}$ as both have the degree $D+1$ (and the later divides the former), is

$$\psi_{\boldsymbol{B}}(x) := \det(x\boldsymbol{I} - \boldsymbol{B}) = \frac{1}{\alpha_D^D} p_{D+1}(x). \tag{35}$$

Let λ_i be an eigenvalue of $\boldsymbol{B}$ with eigenvector $\boldsymbol{v}_i := \boldsymbol{p}(\lambda_i)$, and assume that its multiplicity, as a zero of $\psi_{\boldsymbol{B}}$, is m_i $(1 \leqslant m_i \leqslant m(\lambda_i))$. Then we can repeatedly take the derivative of (32) and evaluate at $x = \lambda_i$ to obtain

$$\begin{aligned}
\boldsymbol{B}\boldsymbol{p}(\lambda_i) &= \lambda_i \boldsymbol{p}(\lambda_i),\\
\boldsymbol{B}\boldsymbol{p}'(\lambda_i) &= \boldsymbol{p}(\lambda_i) + \lambda_i \boldsymbol{p}'(\lambda_i),\\
\boldsymbol{B}\boldsymbol{p}''(\lambda_i) &= 2\boldsymbol{p}'(\lambda_i) + \lambda_i \boldsymbol{p}''(\lambda_i),\\
&\vdots\\
\boldsymbol{B}\boldsymbol{p}^{(m_i-1)}(\lambda_i) &= (m_i-1)\boldsymbol{p}^{(m_i-2)}(\lambda_i) + \lambda_i \boldsymbol{p}^{(m_i-1)}(\lambda_i).
\end{aligned}$$

Then, dividing the κth equation by $(\kappa-1)!$, $1 \leqslant \kappa \leqslant m_i$, and writing all the resulting equalities in matrix form we get

$$\boldsymbol{B}\boldsymbol{p}_i = \boldsymbol{P}_i \boldsymbol{J}_i \quad (0 \leqslant i \leqslant d), \tag{36}$$

where

$$\boldsymbol{P}_i := \left(\boldsymbol{p}(\lambda_i) \left| \frac{\boldsymbol{p}'(\lambda_i)}{1!} \right| \frac{\boldsymbol{p}''(\lambda_i)}{2!} \right| \cdots \left| \frac{\boldsymbol{p}^{(m_i-1)}(\lambda_i)}{(m_i-1)!} \right)$$

is the matrix formed by the m_i columns of $\boldsymbol{P}$ with indices $r_i, r_i+1, \ldots, r_i+m_i-1$ (see the previous subsection) and $\boldsymbol{J}_i$ is the Jordan block of size m_i for the eigenvalue λ_i:

$$\boldsymbol{J}_i := \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

Consequently, we have proved the following result:

Proposition 3.2. *Let Γ be a weakly distance-regular digraph of degree Δ, diameter D, and distance-polynomials $\{p_k\}_{k=0}^{D}$. Let $\boldsymbol{A}$ and $\boldsymbol{B} = (p_{1j}^{k})$ be, respectively, its adjacency and intersection matrices. Then, the following statements hold:*

(i) *The minimum polynomials of $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide with the characteristic polynomial of $\boldsymbol{B}$ which is*

$$\psi_{\boldsymbol{B}} = \frac{1}{\alpha_D^D} p_{D+1} = \frac{1}{\alpha_D^D}(x-\Delta)\sum_{k=0}^{D} p_k,$$

where α_D^D is the leading coefficient of p_D and p_{D+1} is the polynomial defined in (34).

(ii) *If λ_i is an eigenvalue of $\boldsymbol{B}$, then $\boldsymbol{v}_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_D(\lambda_i))^\top$ is a* (*right*) *eigenvector of $\boldsymbol{B}$ corresponding to λ_i.*

(iii) *If $\boldsymbol{B}$ has spectrum $\operatorname{sp}\boldsymbol{B} = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ then it has Jordan normal form $\boldsymbol{J}_{\boldsymbol{B}}$ with blocks $\boldsymbol{J}_0, \boldsymbol{J}_1, \ldots, \boldsymbol{J}_d$ and transformation matrix $\boldsymbol{P}$:*

$$\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{J}_{\boldsymbol{B}}.$$

Notice that the first component of each of the eigenvectors $\boldsymbol{v}_i$ is 1 as $p_0 = 1$. In this case we say that the eigenvector is *standard*. Furthermore, if all the eigenvalues of $\boldsymbol{B}$ are simple then $\{\boldsymbol{v}_i\}_{i=0}^{D}$ is a basis for $\mathbb{C}^{D+1}$ (equivalently, $\boldsymbol{B}$ diagonalizes on $\mathbb{C}$). The following result shows that, in this case, $\boldsymbol{B}$ has also a standard left eigenvector for every λ_i, denoted by $\boldsymbol{u}_i$ (as a column vector), and this allows us to compute the multiplicity of λ_i as an eigenvalue of Γ.

Proposition 3.3. *Let Γ be a weakly distance-regular digraph with order N, diameter D, and intersection matrix $\boldsymbol{B}$. If each eigenvalue λ_i of $\boldsymbol{B}$ is simple, then λ_i has standard left and right eigenvectors, $\boldsymbol{u}_i^\top$ and $\boldsymbol{v}_i$, and the multiplicity of λ_i as an eigenvalue of Γ is*

$$m(\lambda_i) = \frac{N}{\boldsymbol{u}_i^\top \boldsymbol{v}_i} \quad (0 \leqslant i \leqslant d). \tag{37}$$

Proof. Let $p_0 = 1, p_1, \dots, p_D$ be the distance polynomials of Γ. Since all the zeros of m_Γ are simple, then $D = d$ and the (standard) right eigenvectors $\boldsymbol{v}_j$, $0 \leqslant j \leqslant d$, are the columns of the matrix $\boldsymbol{P}_{\mathrm{ev}} = (p_i(\lambda_j))$. Furthermore, since $\{\boldsymbol{v}_j\}_{j=0}^{d}$ is a basis of $\mathbb{C}^{d+1}$, $\boldsymbol{P}_{\mathrm{ev}}$ is invertible, and the rows of $\boldsymbol{P}_{\mathrm{ev}}^{-1}$ are left eigenvectors for $\boldsymbol{B}$. So, from any set $\{\boldsymbol{u}_i\}_{i=0}^{d}$ of such eigenvectors, we can write $\boldsymbol{P}_{\mathrm{ev}}^{-1} = \boldsymbol{D}^{-1}\boldsymbol{U}$, where $\boldsymbol{D} := \mathrm{diag}(\boldsymbol{u}_0^\top \boldsymbol{v}_0, \boldsymbol{u}_1^\top \boldsymbol{v}_1 \dots, \boldsymbol{u}_d^\top \boldsymbol{v}_d)$ and $\boldsymbol{U}$ is the matrix with rows $\boldsymbol{u}_0^\top, \boldsymbol{u}_1^\top, \dots, \boldsymbol{u}_d^\top$. Thus, by Proposition 3.1,

$$m(\lambda_i) = N(\boldsymbol{D}^{-1}\boldsymbol{U})_{i0} = N\frac{u_{i0}}{\boldsymbol{u}_i^\top \boldsymbol{v}_i} \quad (0 \leqslant i \leqslant d).$$

This proves both statements since, as $m(\lambda_i) \neq 0$, it must be that $u_{i0} \neq 0$ and hence we can chose every $\boldsymbol{u}_i$, $0 \leqslant i \leqslant d$, in such a way that $u_{i0} = 1$. (As a by-product, notice also that $\boldsymbol{u}_i^\top \boldsymbol{v}_i$ must be a real number.) $\square$

Here it is worth mentioning the similarity between (37) and the known formula to compute the multiplicities of a distance-regular graph G. Thus, since the corresponding symmetric digraph $\Gamma = G^*$ (obtained by replacing every edge of G by a digon) is distance-regular, the above can be seen as an extension of such a formula to the (more general) directed case. In fact, if Γ is a distance-regular digraph we have, from the orthogonality of the distance polynomials and the simple form of the scalar product (18), that $\boldsymbol{u}_i = (\frac{1}{n_0}\overline{p_0(\lambda_i)}, \frac{1}{n_1}\overline{p_1(\lambda_i)}, \dots, \frac{1}{n_d}\overline{p_d(\lambda_i)})$, where $n_j = n_j(u) = p_j(\Delta)$, $0 \leqslant j \leqslant d$; and the way of computing (37) from such polynomials is the same as for distance-regular graphs (see e.g. [2] or [4]).

4. Some constructions

In this last section we give several constructions and examples of weakly distance-regular digraphs. (For the case of diameter two, we refer the reader to the known constructions of directed strongly regular graphs [5].) For instance, it is shown that every line digraph of a (standard) distance-regular digraph is weakly distance-regular. Moreover two families of digraphs, well-known in the context of network theory, are shown to be weakly distance regular and so they are analyzed in the light of the present theory.

Of course, (trivial) examples of weakly distance-regular digraphs are the known distance-regular digraphs but, for degree $\Delta > 1$, they can only exist for diameter D and girth g satisfying $D \leqslant g \leqslant 8$, if $g > 2$; see [10]. Other easy examples are obtained from the distance-regular (undirected) graphs, which can be seen as distance-regular digraphs with girth two. Indeed, given a distance-regular graph G, its symmetric digraph $\Gamma = G^*$ is clearly a weakly distance-regular digraph.

Let us now see that, through the line digraph technique [16], every distance-regular digraph gives rise to a weakly distance-regular digraph. For a given digraph Γ, its *line digraph* $L\Gamma$ has vertices representing the arcs of Γ, and vertex uv is adjacent to vertex wz when the corresponding arcs are adjacent in Γ; that is, when $v = w$.

Proposition 4.1. *Let Γ be a distance-regular digraph different from a cycle ($\Delta > 1$), with girth g and diameter D. Then its line digraph $L\Gamma$ is a weakly distance-regular digraph. Moreover, if $1, x, p_2, \ldots, p_D$ are the distance polynomials of Γ, then the distance polynomials of $L\Gamma$ are*

$$1, x, xp_1, \ldots, xp_{g-2}, xp_{g-1} - 1, xp_g, \ldots, xp_D.$$

Proof. Let A and $\tilde{A}$ be the adjacency matrices of Γ and $L\Gamma$, respectively. Then, the number of walks of length $l+1$ from vertices $uv \neq wz$ of $L\Gamma$, such that $\mathrm{dist}(uv, wz) = k+1$, $0 \leqslant k \leqslant D$, is

$$\tilde{a}^{(l+1)}_{uv,wz} = a^{(l)}_{v,w} = a^l_k$$

since $\mathrm{dist}(v, w) = k$; whereas in the case $uv = wz$ (that is, $\mathrm{dist}(uv, wz) = 0$) we have

$$\tilde{a}^{(l+1)}_{uv,uv} = a^{(l)}_{v,u} = a^l_{g-1}$$

as $\mathrm{dist}(u, v) = 1$ implies $\mathrm{dist}(v, u) = g - 1$. Thus the number of such walks only depend on the distance between vertices, and $L\Gamma$ is weakly distance-regular.

Moreover, if p_k is the kth distance polynomial of Γ, $0 \leqslant k \leqslant D$, then

$$(\tilde{A} p_k(\tilde{A}))_{uv,wz} = (p_k(A))_{vw} = (A_k)_{vw}.$$

Therefore, taking into account that $(\tilde{A}_{k+1})_{uv,wz} = (A_k)_{vw}$, if $uv \neq wz$, and

$$(\tilde{A}_{k+1})_{uv,uv} = 0 = \begin{cases} (A_k)_{vu} & \text{if } k \neq g-1, \\ (A_k)_{vu} - 1 & \text{if } k = g-1, \end{cases}$$

we conclude that the $(k+1)$th distance polynomial of $L\Gamma$ is xp_k, if $k \neq g-1$, and $xp_{g-1} - 1$, if $k = g-1$. □

In fact, note that the above result still holds under some slightly less restricted condition on Γ. For instance, it suffices to require Γ to be weakly distance-regular, and with every arc (u, v) contained in a cycle of smallest length g (for then

$\mathrm{dist}(v,u)=g-1$). For example, this is the case when $\Gamma=G^*$, G being a distance-regular graph, as mentioned above. More generally, we can consider a weakly distance-regular digraph Γ with every arc a contained in the same number of closed walks of length $l\geqslant 0$. (This is equivalent to require that $L\Gamma$ must be a 'walk regular digraph', a concept defined by Godsil [17] in the undirected case.) As we will see, an example of digraphs satisfying these properties are the so-called butterfly networks, which are the topic of the next subsection.

4.1. The butterfly networks

Butterfly networks have been extensively studied in the literature because of their multiple applications in computer architecture (see e.g. [3,13,18]). In particular, the *wrapped butterfly digraph* $\Gamma=B_\Delta(n)$, of degree Δ and dimension n, has vertices labeled by the pairs $(l,\boldsymbol{x})$, where $l\in\mathbb{Z}_n$ and $\boldsymbol{x}\in\mathbb{Z}_\Delta^n$; and vertex $(l,x_0x_1\dots x_{n-1})$ is adjacent to the vertices $(l+1,x_0\dots x_{l-1}y_lx_{l+1}\dots x_{n-1})$ for any $y_l\in\mathbb{Z}_\Delta$. Thus Γ is a regular digraph with degree Δ, order $N=n\Delta^n$, girth $g=n$, and diameter $D=2n-1$.

Proposition 4.2. *The wrapped butterfly digraph $\Gamma=B_\Delta(n)$ is a weakly distance-regular digraph with distance polynomials*

$$p_0=1,\quad p_1=x,\quad \dots,\quad p_{n-1}=x^{n-1},\quad p_n=x^n-1,$$

$$p_{n+1}=\frac{1}{\Delta}x^{n+1}-x,\quad \dots,\quad p_{2n-1}=\frac{1}{\Delta^{n-1}}x^{2n-1}-x^{n-1}.$$

Furthermore, the eigenvalues of $B_\Delta(n)$ are

$$\lambda_i=\Delta\omega^i\quad (0\leqslant i\leqslant n-1)\quad \textit{and}\quad \lambda_n=0,$$

where ω is any primitive nth root of unity, say, $\omega:=e^{j\frac{2\pi}{n}}$, with respective multiplicities

$$m(\lambda_i)=1\quad (0\leqslant i\leqslant n-1)\quad \textit{and}\quad m(\lambda_n)=N-n.$$

Proof. The invariance of the number of walks, implying the weak distance-regularity, and the expressions for the distance polynomials follow immediately from the following well-known properties of $B_\Delta(n)$: (i) The lengths of all walks from vertex $(l,\boldsymbol{x})$ to vertex $(l',\boldsymbol{y})$ are congruent to $l'-l$ modulo n. (ii) For $0\leqslant k\leqslant n$, the number of vertices at distance k from a given vertex $(l,\boldsymbol{x})$ is the maximum possible for a Δ-regular graph with girth $g=n$, that is, Δ^k if $0\leqslant k\leqslant n-1$ and Δ^n-1 when $k=n$. This means that the (shortest) path from $(l,\boldsymbol{x})$ to each of these vertices is unique. (iii) For $n+1\leqslant k\leqslant 2n-1$, the number of walks of length k from $(l,\boldsymbol{x})$ to every vertex of the form $(l+k,\boldsymbol{y})$ (including those previously reached) is Δ^{k-n}. (To see this, use induction on k.)

Now, from the distance polynomials and Proposition 3.2(i), we know that, apart from $\lambda_0=\Delta$, the distinct eigenvalues of Γ are the zeros of the polynomial

$$S:=\sum\nolimits_{k=0}^{D}p_k=x^n\left(1+\frac{x}{\Delta}+\left(\frac{x}{\Delta}\right)^2+\cdots+\left(\frac{x}{\Delta}\right)^{n-1}\right);$$

that is, $\lambda_i = \Delta\omega^i$, $1 \leqslant i \leqslant n-1$, and $\lambda_n = 0$. Thus,

$$\boldsymbol{P}_{\mathrm{ev}} = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \varDelta & \Delta\omega & \dots & \Delta\omega^{n-1} & 0 \\ \varDelta^2 & \varDelta^2\omega^2 & \dots & \varDelta^2\omega^{(n-1)2} & 0 \\ \vdots & & \ddots & & \vdots \\ \varDelta^n - 1 & \varDelta^n\omega^n - 1 & \dots & \varDelta^n\omega^{(n-1)n} - 1 & -1 \end{pmatrix}$$

has an inverse matrix with first column $\frac{1}{N}(1,1,\dots,1,N-n)^\top$ since $n + (N-n) = N$, $\varDelta^k \sum_{i=0}^{n-1} \omega^{ik} = 0$ for any $1 \leqslant k \leqslant n-1$ and, using that $N = n\varDelta^n$, $\sum_{i=0}^{n-1}(\varDelta^n\omega^{in} - 1) + (N-n)(-1) = n(\varDelta^n - 1) - N + n = 0$. (Of course, this corresponds to verifying Eqs. (20).) Consequently, the multiplicities are as claimed. □

If we allow the presence of multiple arcs, we can reach to the same result in the following way. Let $C_n^\varDelta$ be the directed cycle on n vertices and $\varDelta$ parallel arcs between any adjacent vertices. So, $C_n^\varDelta$ is a weakly distance-regular digraph with girth $g = n$, diameter $D = n-1$, and distance polynomials $p_k = \frac{1}{\varDelta^k} x^k$, $0 \leqslant k \leqslant n-1$. Moreover, for any given $1 \leqslant l \leqslant n$, every walk of length l is contained in a (smallest) cycle of length n. Thus, Proposition 4.1 can be repeatedly applied to prove that the iterated line digraph $L^n C_n^\varDelta$ is a weakly distance-regular digraph with the same parameters as $B_\varDelta(n)$. Then, as both digraphs can be shown to be isomorphic, we get again Proposition 4.2. (To see more details about the relationship between the spectra of a digraph and its line digraph, see [24,22].)

4.2. *The cycle prefix digraphs*

The cycle prefix digraphs were introduced by Faber and Moore as a Cayley coset digraphs, although to study some of their properties it is more useful to consider them as digraphs on alphabets (see [14,7]). The *cycle prefix digraph* $\varGamma = \varGamma_\varDelta(D)$ has vertices of the form $x_1x_2\dots x_D$ where the x_i's are *distinct* elements from the set $\{1,2,\dots,\varDelta+1\}$, and vertex $x_1x_2\dots x_D$ is adjacent to the vertices $x_2\dots x_D x_{D+1}$ and $x_1x_2\dots x_{k-1}x_{k+1}\dots x_D x_k$, $2 \leqslant k \leqslant D-1$. Thus, $\varGamma$ is a $\varDelta$-regular digraph, with order $N = (\varDelta+1)_D \coloneqq \frac{(\varDelta+1)!}{(\varDelta+1-D)!}$ and diameter $D \leqslant \varDelta$. Moreover, in [8,23] the first and last authors showed that the eigenvalues of $\varGamma$ are $\varDelta, D-2, D-1, \dots, 0, -1$. In fact they obtained this result by proving the existence of the distance polynomials, so that the cycle prefix digraph is an example of weakly distance-regular digraph. To show that this is a 'genuine' example, and $\varGamma_\varDelta(D)$ is *not* distance-regular, notice that its girth is $g = 2$ since the digraph has digons. Let us consider the vertices $u = x_1x_2\dots x_D$ and $v = x_2\dots x_D x_1$. Clearly $\operatorname{dist}(u,v) = 1$ but $\operatorname{dist}(v,u) > 1$ if $D > 1$. Hence, the digraph is not stable. (The above can also be proved directly from the definition of strongly distance-regular digraph given in [9].)

Given an integer $x \geqslant 2$, let us denote by $\mathscr{P}_x$ the set of (ordered) t-tuples $(x_1, x_2, \dots, x_t)$, which are a partition of x, $x_1 + x_2 + \dots + x_t = x$, with $x_i \geqslant 2$ for any

$1 \leqslant i \leqslant t$. (Thus, $1 \leqslant t \leqslant \lfloor x/2 \rfloor$.) Now, let Ψ be the function defined on the non-negative integers as

$$\Psi(0) := 1, \quad \Psi(1) := 0, \quad \text{and} \quad \Psi(x) := \sum_{(x_1, x_2, \dots, x_t) \in \mathscr{P}_x} \prod_{i=1}^{t} \frac{(x_i - 2)!}{x_i} \quad (x \geqslant 2).$$

For example, $\Psi(5) = \frac{3!}{5} + \frac{0!}{2}\frac{1!}{3} + \frac{1!}{3}\frac{0!}{2} = \frac{23}{15}$. By looking at the possible values of the first term x_1, we get the following useful recurrence formula for computing Ψ:

$$\Psi(x+1) = \sum_{y=0}^{x-1} \frac{(x-y-1)!}{x-y+1} \Psi(y). \tag{38}$$

In the next result we also use the following notation. For any given integer $k \geqslant 1$, let $(x)_k$ denote the polynomial in x obtained as the product $x(x-1)(x-2)\cdots(x-k+1)$ (note that this definition is consequent with the meaning of $(\Delta+1)_D$).

Theorem 4.3. *The cycle prefix digraph $\Gamma = \Gamma_\Delta(D)$ is a weakly distance-regular digraph with distance polynomials*

$$p_0 = 1, \; p_1 = x, \; p_2 = \frac{(x+1)_3}{x}, \; p_3 = \frac{(x+1)_4}{x-1}, \; \dots, \; p_D = \frac{(x+1)_{D+1}}{x-D+2}.$$

Moreover, the eigenvalues of Γ are

$$\lambda_0 = \Delta \quad \text{and} \quad \lambda_i = D - i - 1 \quad (1 \leqslant i \leqslant D),$$

with respective multiplicities

$$m(\lambda_0) = 1, \; m(\lambda_1) = \frac{1}{(D-1)!}\frac{(\Delta+1)_{D+1}}{\Delta - D + 2}, \; m(\lambda_2) = \frac{1}{(D-2)!}\frac{(\Delta+1)_D}{\Delta - D + 3}, \tag{39}$$

$$m(\lambda_i) = \frac{1}{(D-i)!} \sum_{x=0, x \neq i-2}^{i-1} \Psi(i-1-x) \frac{(\Delta+1)_{D+1-x}}{\Delta - D + 2 + x} \quad (3 \leqslant i \leqslant D-1), \tag{40}$$

$$m(\lambda_D) = (\Delta+1)_D - \sum_{i=0}^{D-1} m(\lambda_i). \tag{41}$$

Proof. As already mentioned, the distance polynomials and the eigenvalues were given in [8]. Indeed, from such polynomials, the eigenvalues are $\lambda_0 = \Delta$ and the zeros of

$$S = \sum_{k=0}^{D} p_k = 1 + \sum_{k=2}^{D+1} \frac{(x+1)_k}{(x-k+3)} = (x+1)_D$$

(to prove the last equality, use induction on D), which gives the above values for λ_i, $1 \leqslant i \leqslant D$. Then, notice that $p_D(\lambda_i) = 0$ for any $i = 2, 3, \dots, D$; and, when $1 \leqslant k \leqslant D-1$, $p_{D-k}(\lambda_i) = 0$ for any $i = k, k+2, k+3, \dots, D$. Consequently, the matrix $\boldsymbol{P}_{\mathrm{ev}}$ has the following quasi-triangular form:

$$\boldsymbol{P}_{\mathrm{ev}} = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \Delta & \lambda_1 & \dots & \lambda_{D-2} & 0 & -1 \\ p_2(\Delta) & p_2(\lambda_1) & \dots & 0 & p_2(\lambda_{D-1}) & \\ p_3(\Delta) & p_3(\lambda_1) & \dots & p_3(\lambda_{D-2}) & & \\ \vdots & & \vdots & & & \\ p_{D-1}(\Delta) & 0 & p_{D-1}(\lambda_2) & & & \\ p_D(\Delta) & p_D(\lambda_1) & & & & \end{pmatrix}$$

which, from (20), allows us to obtain the claimed multiplicities recursively from $m(\Delta) = 1$,

$$m(\lambda_1) = -\frac{p_D(\Delta)}{p_D(\lambda_1)}, \quad m(\lambda_2) = -\frac{p_{D-1}(\Delta)}{p_{D-1}(\lambda_2)}$$

and

$$m(\lambda_i) = \frac{-1}{p_{D-i+1}(\lambda_i)} \sum_{j=0}^{i-1} p_{D-i+1}(\lambda_j) m(\lambda_j) \quad (3 \leqslant i \leqslant D-1)$$

by using (38) (we skip the cumbersome details). Finally, $m(\lambda_D)$ is obtained from the first equation expressing that all multiplicities add up to $N = (\Delta+1)_D$. $\square$

4.3. A possible generalization

We end this section with a digression about a natural generalization of weak distance-regularity and its relation with some of our results. In the last section of Damerell's paper [9], entitled *Related problems*, the author introduce the notion of the so-called weakly distance-transitive digraph. According to him, a digraph Γ is *weakly distance-transitive* if Γ is connected and for any pairs of vertices, (u, v) and (w, z), such that

$$\mathrm{dist}(u, v) = \mathrm{dist}(w, z) \quad \text{and} \quad \mathrm{dist}(v, u) = \mathrm{dist}(z, w),$$

there exists an automorphism $\pi \in \mathrm{Aut}\,\Gamma$ such that $\pi(u) = w$ and $\pi(v) = z$. (Damerell argues that this would be a definition closer to the corresponding concept for graphs). Inspired by this, we could consider an alternative (non-equivalent) definition of a weakly distance-regular digraph Γ as the digraph satisfying the following condition:

(W^*) For any pair of vertices u, v, the number of $u \to v$ walks of every given length $l \geqslant 0$ only depends on the distances $\mathrm{dist}(u, v)$ and $\mathrm{dist}(v, u)$.

Or, in terms of the adjacency matrix $\boldsymbol{A}$ of Γ,

(W^*) Every matrix power $\boldsymbol{A}^l$, $l \geqslant 0$, is a linear combination of the Hadamard products of distance matrices $\boldsymbol{A}_i \circ \boldsymbol{A}_j^{\top}$, $0 \leqslant i, j \leqslant D$.

Observe that condition (W^*) is stronger than the property characterizing walk regular digraphs, but 'weaker' that our condition characterizing weakly distance-regularity. Namely,

(W) For any pair of vertices u, v, the number of $u \to v$ walks of every given length $l = 0, 1, \ldots, D$ only depend on the distance $\mathrm{dist}(u, v)$.

In this context we have the following result (we omit the proof, as some of the equivalences have been already given):

Theorem 4.4. *Let $\Gamma = (V, E)$ be a digraph with adjacency matrix $\boldsymbol{A}$ and girth g. Then the following statements are equivalent*:

(a) *Γ is distance-regular.*
(b) *Γ satisfies condition* (W^*) *and it is stable.*
(c) *Γ satisfies condition* (W) *and, for any pair of vertices $u, v \in V$, we have* $\mathrm{dist}(u, v) = 1 \Leftrightarrow \mathrm{dist}(v, u) = g - 1$ (*or, equivalently,* $\boldsymbol{A}^\top = \boldsymbol{A}_{g-1}$).
(d) *Γ satisfies condition* (W) *and $\boldsymbol{A}$ is normal.*

One of the referees pointed out that, recently, Suzuki and Wang [27] suggested that a 'weakly distance-regular digraph' is a digraph with the following property: for vertices u and v such that $\mathrm{dist}(u, v) = k_1$ and $\mathrm{dist}(v, u) = k_2$, the number of vertices z satisfying $\mathrm{dist}(u, z) = i_1$, $\mathrm{dist}(z, u) = i_2$, $\mathrm{dist}(v, z) = j_1$, and $\mathrm{dist}(z, v) = j_2$ depend only on the values $k_1, k_2, i_1, i_2, j_1, j_2$. (This is equivalent to require that the Hadamard products $A_i \circ A_j^\top$ form an association scheme.) Weakly distance-transitive digraphs satisfy this property, but it is not clear whether these digraphs are easily related with property (W^*).

Acknowledgments

The authors are indebted to the referees for helpful comments and suggestions which lead to numerous improvements of the manuscript.

References

[1] E. Bannai, P.J. Cameron, J. Kahn, Nonexistence of certain distance-transitive digraphs, J. Combin. Theory Ser. B 31 (1981) 105–110.
[2] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, CA, 1984.
[3] J.-C. Bermond, E. Darrot, O. Delmas, S. Perennes, Hamilton circuits in the directed wrapped butterfly network, Discrete Appl. Math. 84 (1–3) (1998) 21–42.
[4] N. Biggs, Algebraic Graph Theory, Cambridge University Press, Cambridge, 1974; 2nd Edition, 1993.
[5] A. Brouwer, personal homepage: http://www.cwi.nl/~aeb/math/dsrg/dsrg.html.

[6] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, Cambridge, 1991.
[7] F. Comellas, M.A. Fiol, Vertex-symmetric digraphs with small diameter, Discrete Appl. Math. 58 (1) (1995) 1–12.
[8] F. Comellas, M. Mitjana, The spectra of cycle prefix digraphs, SIAM J. Discrete Math. 16 (3) (2003) 418–421.
[9] R.M. Damerell, Distance-transitive and distance-regular digraphs, J. Combin. Theory Ser. B 31 (1) (1981) 46–53.
[10] D.A. Douglas, K. Nomura, The girth of a directed distance-regular graph, J. Combin. Theory Ser. B 58 (1) (1993) 34–39.
[11] A.M. Duval, A directed graph version of strongly regular graphs, J. Combin. Theory Ser. A 47 (1) (1988) 71–100.
[12] H. Enomoto, R.A. Mena, Distance-regular digraphs of girth 4, J. Combin. Theory Ser. B 43 (3) (1987) 293–302.
[13] M. Espona, O. Serra, Cayley digraphs based on the de Bruijn networks, SIAM J. Discrete Math. 11 (2) (1998) 305–317.
[14] V. Faber, J.W. Moore, W.Y.C. Chen, Cycle prefix digraphs for symmetric interconnection networks, Networks 23 (1993) 641–649.
[15] M.A. Fiol, Algebraic characterizations of distance-regular graphs, Discrete Math. 246 (1–3) (2002) 111–129.
[16] M.A. Fiol, J.L.A. Yebra, I. Alegre, Line digraphs iterations and the (d,k) digraph problem, IEEE Trans. Comput. C-32 (1984) 400–403.
[17] C.D. Godsil, Algebraic Combinatorics, Chapman & Hall, New York, 1993.
[18] R. Klasing, R. Lüling, B. Monien, Compressing cube-connected cycles and butterfly networks, Networks 32 (1) (1998) 47–65.
[19] C.W.H. Lam, Distance-transitive digraphs, Discrete Math. 29 (3) (1980) 265–274.
[20] P. Lancaster, M. Tismenetsky, The Theory of Matrices, 2nd Edition, Academic Press, Orlando, FL, 1985.
[21] R.A. Liebler, R.A. Mena, Certain distance-regular digraphs and related rings of characteristic 4, J. Combin. Theory Ser. A 47 (1) (1988) 111–123.
[22] X. Marcote, C. Balbuena, I. Pelayo, The relationship between the Jordan normal forms of a digraph and its line digraph, Research Report, Universitat Politècnica de Catalunya, 1999.
[23] M. Mitjana, Propagació d'informació en grafs i digrafs que modelen xarxes d'interconnexió simètriques (Catalan), Ph.D. Thesis, Universitat Politècnica de Catalunya, 1999.
[24] J.C. Montserrat, Propiedades matriciales de los grafos dirigidos, Master Thesis, Universitat Politècnica de Catalunya, 1986 (Spanish).
[25] P. Rowlinson, Linear algebra, in: L.W. Beineke, R.J. Wilson (Eds.), Graph Connections, Oxford Lecture Series in Mathematical Applications, Vol. 5, Oxford University Press, New York, 1997, pp. 86–99.
[26] T. Takahashi, Distance-regular digraphs of girth 6, Mem. Fac. Sci. Kyushu Univ. Ser. A 45 (2) (1991) 155–166.
[27] K. Wang, H. Suzuki, Weakly distance-regular digraphs, Discrete Math. 264 (1–3) (2003) 225–263.